

A Pieri-Chevalley formula for $K(G/B)$

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0. Introduction

Algebraic combinatorics and geometry come together in a beautiful way in the study of the cohomology $H^*(G/B)$ of the generalized flag variety G/B for a complex semisimple Lie group G with Borel subgroup B . The cohomology $H^*(G/B)$ is isomorphic (as a graded ring) to the quotient of a polynomial ring by the ideal generated by W -symmetric functions without constant term and has a natural basis of “Schubert polynomials” $[X_w] \in H^*(G/B)$ where $[X_w]$ is the Poincaré dual of the fundamental class of the Schubert variety $X_w \subseteq G/B$ in $H_*(G/B)$. One of the fundamental results in the theory of Schubert polynomials and the cohomology of the flag variety is a formula of Chevalley which gives an expansion of the product $\lambda \cdot [X_w]$ in terms of the Schubert class basis for an element $\lambda \in H^2(G/B)$.

A similar picture holds for the K-theory of G/B . The ring $K(G/B)$ is isomorphic to a quotient of a *Laurent* polynomial ring by an ideal generated by certain W -symmetric functions and has a basis given by classes $[\mathcal{O}_{X_w}]$ where \mathcal{O}_{X_w} is the structure sheaf of the Schubert variety $X_w \subseteq G/B$ extended by 0 outside of X_w . In this paper we give an analogue of Chevalley’s formula for the ring $K(G/B)$. Specifically, we give an explicit combinatorial formula for $e^\lambda[\mathcal{O}_{X_w}]$, the tensor product of a (negative) line bundle with the structure sheaf of a Schubert variety, expanded in terms of the Schubert class basis $\{[\mathcal{O}_{X_w}]\}$.

The Chern character is an isomorphism between $K(G/B)$ and $H^*(G/B)$ and Chevalley’s formula can be recovered from ours by applying the Chern character and comparing lowest degree terms. The higher order terms of our formula may yield further interesting identities in cohomology.

Fulton and Lascoux [FL] have given a formula similar to ours for the $G = GL_n(\mathbb{C})$ case. Our formula is a generalization of their formula to general type except that we work only with $K(G/B)$ instead of the K-theory of the flag bundle. In our work the column strict tableaux used by Fulton and Lascoux are replaced by Littelmann’s path model. This has two advantages: (1) it allows us to work in general type and (2) it obviates the need for the complex combinatorics associated with the jeu de taquin and the “rectification” of tableaux. It is possible that the more general

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flag bundle result of Fulton and Lascoux can be obtained from our general commutation formula, Theorem 4.2, but to sort this out properly one would have to understand concretely the connection between the tableaux and the Littelmann paths.

In the first section of the paper we review notations and recall why one is able to work with $K(G/B)$ as a quotient of a Laurent polynomial ring. In the second section we derive an expression for $[\mathcal{O}_p] \in K(G/P)$, for a point $p \in G/P$, in terms of familiar vector bundles. We are able to pull back this formula into $K(G/B)$ to obtain formulas for certain special Schubert classes in terms of line bundles. In section 3 we recall the operators which play the same role as the BGG operators in cohomology and show that they can be used to give explicit (inductive) expressions for the Schubert classes in $K(G/B)$. In section 4 we prove the main theorem, which gives a commutation relation between line bundles and the Schubert classes. Our new Pieri-Chevalley formula is an immediate consequence of this relation. In the final section we explain how the K-theory relates to cohomology and how our formula implies the classical Chevalley formula.

The main results of the preliminary sections can all be considered well known. These results can be found, either explicitly or implicitly, in the work of Demazure [D], Kostant and Kumar [KK], Fulton and Lascoux [FL] and others. For the convenience of the reader, we have given short proofs or sketches of proofs for most of these results and, in the final section, we have given a dictionary between K-theory and $H^*(G/B)$. This dictionary illustrates how our results relate to the theory of Schubert polynomials.

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1. Background

In this section we set up notations and recall how one is able to work with the K-theory of G/B “purely combinatorially”. This is possible because (in analogy with the cohomology $H^*(G/B)$ and the combinatorial theory of Schubert polynomials) the K-theory of G/B is isomorphic to a quotient of $R(T)$, which is a ring of Laurent polynomials. In section 5 we shall explain how our results about $K(G/B)$ relate to and generalize well known results about Schubert polynomials and $H^*(G/B)$.

The K-theory $K(G/B)$

If X is a quasi-projective variety let

$$\begin{aligned} K(X) &= \text{the Grothendieck group of coherent sheaves on } X, \\ K_{lf}(X) &= \text{the Grothendieck group of locally free sheaves on } X, \\ K_{vb}(X) &= \text{the Grothendieck group of vector bundles on } X. \end{aligned}$$

If $f: X \rightarrow Y$ is a morphism of projective varieties then we have maps

$$\begin{aligned} f^!: K(Y) &\longrightarrow K(X) \\ [\mathcal{F}] &\longmapsto [f^*\mathcal{F}], \\ f_!: K(X) &\longrightarrow K(Y) \\ [\mathcal{F}] &\longmapsto \sum_i (-1)^i [R^i f_* \mathcal{F}]. \end{aligned}$$

See [CG, §5.2] for K-theory background. If X is smooth then the isomorphism between $K(X)$ and $K_{lf}(X)$ is given by assigning to the class of a sheaf the alternating sum of the sheaves in a locally

free resolution, see [BS, §4] and [Ha, III Ex. 6.9]. There is always a map $K_{lf}(X) \rightarrow K_{vb}(X)$ which assigns to a locally free sheaf the underlying vector bundle and the results of [P] imply that this map is an isomorphism when $X = G/B$, see Proposition 1.5 below. Thus, in the case which we wish to consider in this paper, $X = G/B$, all three K -theories are isomorphic.

Let G be a complex connected simply connected semisimple Lie group. Fix a maximal torus T and a Borel subgroup B such that $T \subseteq B \subseteq G$. The Bruhat decomposition says that G is a disjoint union of double cosets of B indexed by the elements of the Weyl group W , $G = \bigcup_{w \in W} BwB$. The *flag variety* is the projective variety formed by the coset space G/B and the Bruhat decomposition of G induces a cell decomposition of G/B . For each $w \in W$ the subset $X_w^\circ = BwB/B$ is the *Schubert cell* and its closure X_w is the *Schubert variety*. The formulas

$$\dim(X_w^\circ) = \ell(w) \quad \text{and} \quad X_w = \bigcup_{v \leq w} X_v^\circ \quad (1.1)$$

define the *length* $\ell(w)$ of $w \in W$ and the *Bruhat-Chevalley order* \leq on the Weyl group, respectively. It follows from the Bruhat decomposition (see lecture 4 by Grothendieck in [C]) that

$$K(G/B) \text{ is a free } \mathbb{Z}\text{-module with basis } \{[\mathcal{O}_{X_w}] \mid w \in W\},$$

where X_w are the Schubert varieties in G/B and \mathcal{O}_{X_w} is the structure sheaf of X_w extended to G/B by defining it to be 0 outside X_w .

The isomorphism $K(G/B) \cong R(T)/\mathcal{I}$

For any group H let $R(H)$ be the Grothendieck group of complex representations of H . Let $\Lambda = \sum_{i=1}^n \mathbb{Z}\omega_i$, where the ω_i are the fundamental weights of the Lie algebra \mathfrak{g} of G . We shall use the “geometric” convention (see [CG, 6.1.9(ii)]) and let $e^{-\lambda}$ be the element of $R(T)$ corresponding to the character determined by $\lambda \in \Lambda$. Then

$$R(T) \text{ has } \mathbb{Z}\text{-basis } \{e^\lambda \mid \lambda \in \Lambda\}, \quad \text{with multiplication } e^\lambda e^\mu = e^{\lambda+\mu},$$

and Weyl group action determined by $we^\lambda = e^{w\lambda}$, for $w \in W$ and $\lambda \in \Lambda$. In this way $R(T)$ is a Laurent polynomial ring and $R(G) \cong R(T)^W$ is the subalgebra of “symmetric functions” in $R(T)$.

Suppose that V is a T -module. Since $T \cong B/U$, where U is the unipotent radical of B , we can extend V to be a B -module by defining the action of U to be trivial. Define a vector bundle

$$\pi: \begin{array}{ccc} G \times_B V & \longrightarrow & G/B \\ (g, v) & \longmapsto & gB \end{array} \quad \text{where} \quad G \times_B V = \frac{G \times V}{\langle (g, v) \sim (gb, b^{-1}v) \rangle},$$

so that $G \times_B V$ is the set of pairs (g, v) , $g \in G$, $v \in V$, modulo the equivalence relation $(g, v) \sim (gb, b^{-1}v)$. This construction induces a ring homomorphism

$$\phi: \begin{array}{ccc} R(T) & \longrightarrow & K(G/B) \\ V & \longmapsto & (G \times_B V \xrightarrow{\pi} G/B). \end{array} \quad (1.2)$$

If V is a G -module then $\phi(V) = \dim(V)$ in $K(G/B)$. This is because the map

$$\begin{array}{ccc} G \times_B V & \longrightarrow & G/B \times V \\ (g, x) & \longmapsto & (gB, gx) \end{array} \quad (1.3)$$

is an isomorphism between $G \times_B V$ and the trivial bundle $G/B \times V$. Define $\varepsilon: R(T) \rightarrow \mathbb{Z}$ by $\varepsilon(e^\lambda) = 1$ for $\lambda \in \Lambda$. Then the map ϕ in (1.2) gives an isomorphism (see Proposition 1.5 below)

$$K(G/B) \cong R(T)/\mathcal{I},$$

where \mathcal{I} is the ideal generated by $\{f \in R(T)^W \mid f - \varepsilon(f) = 0\}$. Equivalently, $K(G/B) \cong R(T) \otimes_{R(G)} \mathbb{Z}$, where $R(G)$ acts on \mathbb{Z} by $[V] \cdot 1 = \dim(V)$, if V is a G -module.

$K(G/P)$ for a parabolic subgroup P

A similar setup works when B is replaced by any parabolic subgroup P containing B . The coset space G/P is a projective variety and the Bruhat decomposition takes the form $G = \bigcup_{\bar{w} \in W/W_P} B\bar{w}P$ where W_P is the subgroup of W given by $W_P = \langle s_i \mid \mathfrak{g}_{-\alpha_i} \in \mathfrak{p} \rangle$, where \mathfrak{p} is the Lie algebra of P . The Schubert varieties $X_{\bar{w}}$ are the closures of the Schubert cells $X_{\bar{w}}^\circ = B\bar{w}P$ in G/P .

$K(G/P)$ is a free \mathbb{Z} -module with basis $\{[\mathcal{O}_{X_{\bar{w}}}] \mid \bar{w} \in W/W_P\}$.

Write $P = LU$ where U is the unipotent radical of P and L is a Levi subgroup. The Weyl group of L is W_P and

$$R(L) \cong R(T)^{W_P}$$

is the subring of W_P -symmetric functions in $R(T)$. The same construction as in (1.2) with B replaced by P and T replaced by L gives a ring homomorphism

$$\begin{aligned} \phi_P: R(L) &\longrightarrow K(G/P) \\ V &\longmapsto (G \times_P V \xrightarrow{\pi} G/P). \end{aligned} \quad (1.4)$$

Proposition 1.5. *Let G be a connected simply connected semisimple Lie group and let T be a maximal torus of G . Let P be a parabolic subgroup of G with Levi decomposition $P = LU$ and let W_P be the Weyl group of L . Then*

$$K(G/P) \cong \frac{R(T)^{W_P}}{\mathcal{I}_P},$$

where \mathcal{I}_P is the ideal generated by $\{f \in R(T)^{W_P} \mid f - \varepsilon(f) = 0\}$ and $\varepsilon: R(T) \rightarrow \mathbb{Z}$ is the map given by $\varepsilon(e^\lambda) = 1$ for $\lambda \in \Lambda$. Equivalently, $K(G/P) = R(L) \otimes_{R(G)} \mathbb{Z}$.

Proof. Let $K_{vb}(G/P)$ be the Grothendieck group of C^∞ vector bundles on G/P and let $\eta: K(G/P) \rightarrow K_{vb}(G/P)$ be the map which assigns to a locally free sheaf its underlying vector bundle. Let ϕ_P be the composition

$$\tilde{\phi}_P: R(L) \xrightarrow{\phi_P} K(G/P) \xrightarrow{\eta} K_{vb}(G/P).$$

Since $\pi_1(G) = 0$, $\pi_1(P) \cong \pi_2(G/P)$, which is free abelian by the Bruhat decomposition. Since the unipotent radical U of P is contractible the projection $f: P \rightarrow P/U \cong L$ is a homotopy equivalence. Thus $\pi_1(L)$ is free abelian and we may apply the results of [P] to conclude that $\tilde{\phi}_P: R(L) \rightarrow K_{vb}(G/P)$ is surjective, $R(L)$ is projective over $R(G)$ with rank $|W/W_P|$ and $K_{vb}(G/P)$ is a free \mathbb{Z} -module of the same rank. (Note: The results of [P] can be applied since G and L are the complexifications of compact groups.)

Since $\tilde{\phi}_P$ is surjective the map η is also surjective. Then, since $K(G/P)$ and $K_{vb}(G/P)$ are both free \mathbb{Z} -modules of rank $|W/W_P|$, it follows that η must be an isomorphism. This means two things: (1) that we can identify $K(G/P)$ and $K_{vb}(G/P)$, and (2) that ϕ_P is surjective.

The kernel \mathcal{I}_P of ϕ_P is identified by using (1.3). ■

Transfer from $K(G/B)$ to $K(G/P)$

Although we will work primarily with $K(G/B)$ it is standard to transfer results from $K(G/B)$ to results on $K(G/P)$. This can be accomplished with the following proposition. The proof will be given in section 5.

Proposition 1.6. *If $f: G/B \rightarrow G/P$ is the natural projection then the induced map $f^!: K(G/P) \rightarrow K(G/B)$ is an injection. This map is given explicitly by*

$$f^!([\mathcal{O}_{X_{\bar{w}}}]) = [\mathcal{O}_{X_v}],$$

where $v \in W$ is the unique element of longest length in the coset $\bar{w} = vW_P$.

2. The class $[\mathcal{O}_{P/B}]$ in $K(G/B)$

In this section we give an expression for the class $[\mathcal{O}_{P/B}] \in K(G/B)$ as an element of $R(T)/\mathcal{I}$. This is done by first finding a formula for $[\mathcal{O}_{P/P}]$ in $K(G/P)$ and then using the projection $f: G/B \rightarrow G/P$ to pull back this formula to $K(G/B)$. The formula for $[\mathcal{O}_{P/P}]$ in $K(G/P)$ is obtained by using a Koszul resolution on a vector field with simple zeros at the points $\{\bar{w}_i P \mid \bar{w}_i \in W/W_P\}$. This reduces the computation to determining $\Lambda_{-1}(T^*(G/P))$ and this can be done since we understand the structure of $T^*(G/P) = (\mathfrak{g}/\mathfrak{p})^*$ as a B -module (under the adjoint action). Although these formulae for $[\mathcal{O}_{P/B}]$ are useful for specific computations they are not needed for the proof of our main result, Theorem 4.3.

Theorem 2.1. *Let $P \supseteq B$ be a parabolic subgroup of G and let w be the longest element of the corresponding parabolic subgroup W_P of W . In $K(G/B)$*

$$[\mathcal{O}_{X_w}] = \frac{|W_P|}{|W|} \prod_{\mathfrak{g}-\alpha \notin \mathfrak{p}} (1 - e^{-\alpha}),$$

where the product is over all positive roots α such that $\mathfrak{g}_{-\alpha} \notin \mathfrak{p}$.

Proof. Let \mathfrak{h} denote the complex Lie algebra of the maximal torus $T \subseteq G$ and let $H \in \mathfrak{h}$ be a regular element, i.e. the W action on H has trivial stabilizer. The one-parameter group $\exp(zH)$, $z \in \mathbb{C}$, of G induces a flow on G/P (by left translation) whose fixed points are the points in the set

$$Z = \{w_i P \in G/P\},$$

where w_i run over a set of coset representatives of W/W_P . It follows that the zeros of the associated vector field $v(H)$ are the same points and a local calculation shows that they are simple. This construction of vector fields $v: G/P \rightarrow T(G/P)$ whose zeros are isolated and simple is essentially due to A. Weil [W].

Since the zero set Z of the vector field $v(H)$ is a smooth subvariety of codimension equal to the fibre dimension of $T(G/P)$, the vector field $v(H)$ gives rise to a Koszul resolution of \mathcal{O}_Z (see [CG] §5.4)

$$\cdots \xrightarrow{i_v} \mathcal{O}_{G/P}(\wedge^2(T^*(G/P))) \xrightarrow{i_v} \mathcal{O}_{G/P}(\wedge^1(T^*(G/P))) \longrightarrow \mathcal{O}_{G/P} \longrightarrow \mathcal{O}_Z \longrightarrow 0,$$

where i_v denotes interior product with $v(H)$. Hence, in $K(G/P)$ we have

$$[\mathcal{O}_Z] = \Lambda_{-1}(T^*(G/P)) \quad \text{where} \quad \Lambda_{-1}(T^*(G/P)) = \sum_i (-1)^i \wedge^i(T^*(G/P)).$$

For any two points $p, q \in G/P$ there is a $g \in G$ so that $gp = q$. Since G acts trivially on $K(G/P)$,

$$[\mathcal{O}_q] = g^![\mathcal{O}_p] = [g^*\mathcal{O}_p] = [\mathcal{O}_p]. \quad (2.2)$$

Since the points $\bar{w}_i P$ of Z are simple,

$$\mathcal{O}_Z = \bigoplus_{i=1}^{|W/W_P|} \mathcal{O}_{\bar{w}_i P} \quad \text{and so} \quad [\mathcal{O}_Z] = \sum_{i=1}^{|W/W_P|} [\mathcal{O}_{\bar{w}_i P}] = |W/W_P|[\mathcal{O}_P],$$

by (2.2). Since $K(G/P)$ is a free \mathbb{Z} -module it follows that $[\mathcal{O}_Z]$ is divisible by $|W/W_P|$ and we get

$$[\mathcal{O}_P] = \frac{|W_P|}{|W|} \Lambda_{-1}(T^*(G/P)). \quad (2.3)$$

Let us compute the pull back $f^! (|W/W_P|[\mathcal{O}_P]) = f^! (\Lambda_{-1}(T^*(G/P))) \in K(G/B)$ for the projection $f: G/B \rightarrow G/P$. The bundle $T^*(G/P)$ is the homogeneous vector bundle over G/P associated to the P -module $(\mathfrak{g}/\mathfrak{p})^*$, where P acts on $\mathfrak{g}/\mathfrak{p}$ by the adjoint action. Then $f^!(T^*(G/P))$ is the vector bundle over G/B associated to the B -module $(\mathfrak{g}/\mathfrak{p})^*$, where we regard $(\mathfrak{g}/\mathfrak{p})^*$ as a B -module by restriction. By Lie's theorem, $(\mathfrak{g}/\mathfrak{p})^*$ admits an B -module filtration such that the unipotent radical of B acts trivially on the associated graded module $\text{gr}_F(\mathfrak{g}/\mathfrak{p})^*$. Hence

$$\text{gr}_F(\mathfrak{g}/\mathfrak{p})^* = \sum_{\mathfrak{g}-\alpha \notin \mathfrak{p}} \mathfrak{g}_\alpha$$

is a sum of weight spaces as an $\text{ad}(B)$ -module. Since a filtered object and its associated graded define the same element in a Grothendieck ring we have

$$f^![T^*(G/P)] = \sum_{\mathfrak{g}-\alpha \notin \mathfrak{p}} e^{-\alpha}$$

in $K(G/B)$. From this equation we get the formula for

$$f^!(\Lambda_{-1}(T^*(G/P))) = \prod_{\mathfrak{g}-\alpha \notin \mathfrak{p}} (1 - e^{-\alpha}). \quad (2.4)$$

The theorem follows from (2.3), (2.4) and Proposition 1.6 since

$$f^!([\mathcal{O}_P]) = [\mathcal{O}_{f^{-1}(P)}] = [\mathcal{O}_{P/B}]$$

and $P/B = X_w$ for the longest element w of $W_P \subseteq W$. ■

Corollary 2.5. *In $K(G/B)$*

$$\begin{aligned} [\mathcal{O}_{X_1}] &= \frac{1}{|W|} \prod_{\alpha > 0} (1 - e^{-\alpha}), \\ [\mathcal{O}_{X_{s_i}}] &= \frac{2}{|W|} \prod_{\substack{\alpha > 0 \\ \alpha \neq \alpha_i}} (1 - e^{-\alpha}), \quad \text{for each simple reflection } s_i, 1 \leq i \leq n, \text{ and} \\ [\mathcal{O}_{X_{w_0}}] &= 1, \quad \text{for the longest element } w_0 \text{ in } W. \end{aligned}$$

Proof. The first and last formulas are Theorem 2.1 in the cases $P = B$ and $P = G$ respectively and the middle formula is the case when $P = P_i$ is the minimal parabolic subgroup whose Lie algebra \mathfrak{p}_i is generated by \mathfrak{b} and the negative root space $\mathfrak{g}_{-\alpha_i}$. ■

Remarks.

1. In optimal cases such as $G = SL(n, \mathbb{C})$ one can use various tautological bundles on $SL(n, \mathbb{C})/P$ to construct resolutions of \mathcal{O}_p directly, and hence obtain formulae for $[\mathcal{O}_p]$ which are “denominator free”. One example is obtained from the tautological k -plane bundle over the Grassmannian of k -planes in \mathbb{C}^n : $E_k \rightarrow \mathbb{G}(k, \mathbb{C}^n)$. Every (homogeneous) linear function on \mathbb{C}^n defines an algebraic section of E_k^* . Hence by choosing $(n-k)$ linearly independent such functions, we can define a section $\sigma: \mathbb{G}(k, \mathbb{C}^n) \rightarrow \bigoplus_{n-k} E_k^*$ whose unique zero is the point p corresponding to the common kernel of the linear functions. Since σ is clearly regular, $[\mathcal{O}_p] = \sum (-1)^i [\wedge^i (\bigoplus_{n-k} E_k)]$ in $K(\mathbb{G}(k, \mathbb{C}^n))$. Other examples can be found in [FL].

2. In contrast with the previous remark, it seems difficult to find “denominator free” formulae for $[\mathcal{O}_p]$ in general. A comparison with cohomology will be helpful. For $\mathbb{F}(\mathbb{C}^n) = SL(n, \mathbb{C})/B$ a generator of the top cohomology is given by $x_1^{n-1} x_2^{n-2} \cdots x_{n-1}$, where $x_j \in H^2(\mathbb{F}(\mathbb{C}^n); \mathbb{Z})$ form a suitable basis. For general G/B the only uniform expressions for a generator in the top degree all involve denominators. For example, one such is $\frac{1}{|W|} \prod_{\alpha > 0} \alpha$. Indeed, if $H^*(G; \mathbb{Z})$ has torsion then no integral polynomial in a basis for $H^2(G/B; \mathbb{Z})$ will give a generator in the top degree.

3. Push-pull operators in K-theory

For a positive root α , let $s_\alpha \in W$ be the corresponding reflection and define operators $L_\alpha: R(T) \rightarrow R(T)$ and by $T_\alpha: R(T) \rightarrow R(T)$ by

$$L_\alpha(x) = \frac{x - s_\alpha x}{1 - e^{-\alpha}} \quad \text{and} \quad T_\alpha(x) = e^{-\rho} L_\alpha(e^\rho x) = \frac{e^\alpha x - s_\alpha x}{e^\alpha - 1},$$

respectively, where $\rho = \frac{1}{2} \sum_{\alpha > 0} \alpha$. In this section we will show that there is an inductive formula for the classes $[\mathcal{O}_{X_w}]$ in terms of the operators T_α and the class $[\mathcal{O}_{X_1}]$, which was determined in Corollary 2.5.

The operators T_α and L_α have been in the literature for some time, see for example, [D, §5], [KK], [FL]. Let $x, y \in R(T)$. Short direct calculations using the definitions establish the following identities:

$$(3.1a) \quad T_\alpha(xy) = xT_\alpha(y), \quad \text{if } s_\alpha x = x,$$

$$(3.1b) \quad s_\alpha T_\alpha x = T_\alpha x,$$

$$(3.1c) \quad T_\alpha T_\alpha x = T_\alpha x,$$

$$(3.1d) \quad e^\lambda T_\alpha x = \left(T_\alpha e^{s_\alpha \lambda} + \frac{e^\lambda - e^{s_\alpha \lambda}}{1 - e^{-\alpha}} \right) x.$$

Because of (a), T_α is a map of $R(G)$ -modules and so it descends to an operator on $K(G/B)$ which we shall denote by the same symbol. Moreover, the induced operator on $K(G/B)$ satisfies (3.1a-d).

Let α be a simple root and let P_α be the minimal parabolic subgroup whose Lie algebra \mathfrak{p}_α is generated by \mathfrak{b} and the negative root space $\mathfrak{g}_{-\alpha}$. Since $P_\alpha/B \cong \mathbb{P}_1$, the natural projection

$$f_\alpha: G/B \rightarrow G/P_\alpha \tag{3.2}$$

is a \mathbb{P}_1 -bundle.

Proposition 3.3. *Let α be a simple root. For every $x \in K(G/B)$,*

$$(f_\alpha)^! \circ (f_\alpha)_! (x) = T_\alpha(x).$$

This result is proved in [KK, Prop. 4.11]. In section 5 we shall see that the Grothendieck-Riemann-Roch theorem implies that this fact is equivalent to the corresponding fact in cohomology. This alternate point of view has the advantage that it illustrates why the operators T_α are the K-theoretic analogues of the BGG operators ∂_α (see [BGG] and [D]). The proof of following proposition is a generalization of the argument in [FL, p. 728]. Kostant and Kumar [KK, Lemma 4.12] have also proved the same result.

Proposition 3.4. *Let $s_\alpha \in W$ be the simple reflection corresponding to a simple root α . Given a Schubert variety $X_w \subseteq G/B$,*

$$(f_\alpha)^! \circ (f_\alpha)_! ([\mathcal{O}_{X_w}]) = \begin{cases} [\mathcal{O}_{X_{ws_\alpha}}], & \text{if } \ell(ws_\alpha) > \ell(w), \\ [\mathcal{O}_{X_w}], & \text{if } \ell(ws_\alpha) < \ell(w). \end{cases}$$

Proof. The main idea of the proof is

$$(f_\alpha)^! \circ (f_\alpha)_! ([\mathcal{O}_{X_w}]) = (f_\alpha)^! ([\mathcal{O}_{f_\alpha(X_w)}]) = [\mathcal{O}_{f_\alpha^{-1}(f_\alpha(X_w))}].$$

One only has to justify the equalities and identify $f_\alpha(X_w)$ and $f_\alpha^{-1}(f_\alpha(X_w))$.

For $w \in W$ let $\bar{w} = \{w, ws_\alpha\}$. It is convenient to relabel the elements of the set $\{w, ws_\alpha\}$ as w' and w'' where by fiat $\ell(w'') = \ell(w') + 1$. Analyzing the Bruhat decomposition of X_w in (1.1) we get

$$f_\alpha(X_{w'}) = f_\alpha(X_{w''}) = X_{\bar{w}} \quad \text{and} \quad f_\alpha^{-1}(X_{\bar{w}}) = X_{w''}. \quad (3.5)$$

Since $f_\alpha: X_{w'}^\circ \rightarrow X_{\bar{w}}^\circ$ is an isomorphism of varieties $f_\alpha: X_{w'} \rightarrow X_{\bar{w}}$ is birational. This combined with the (deep) fact that Schubert varieties have rational singularities (see the survey [Ra] and the references there) implies that

$$(a) \quad (f_\alpha)_* (\mathcal{O}_{X_{w'}}) = \mathcal{O}_{X_{\bar{w}}}, \text{ and } R^q(f_\alpha)_* (\mathcal{O}_{X_{w'}}) = 0, \text{ for } q > 0.$$

From the Bruhat decomposition one sees that $f_\alpha: X_{w''} \rightarrow X_{\bar{w}}$ is the restriction of the ambient \mathbb{P}_1 -bundle $f_\alpha: G/B \rightarrow G/P_\alpha$. Thus

$$(b) \quad (f_\alpha)_* (\mathcal{O}_{X_{w''}}) = \mathcal{O}_{X_{\bar{w}}} \text{ and } R^q(f_\alpha)_* (\mathcal{O}_{X_{w''}}) = 0, \text{ for } q > 0.$$

Finally, from (3.5) we have

$$(c) \quad (f_\alpha)^* (\mathcal{O}_{X_{\bar{w}}}) = \mathcal{O}_{X_{w''}}.$$

Statements (a) and (b) imply that $(f_\alpha)_! ([\mathcal{O}_{X_w}]) = [\mathcal{O}_{f_\alpha(X_w)}]$ and (c) implies that $(f_\alpha)^! ([\mathcal{O}_{X_{\bar{w}}})) = [\mathcal{O}_{X_{w''}}]$. ■

Corollary 3.6. *For each simple root α_i let $T_i = T_{\alpha_i}$. Let $w = s_{i_1} \cdots s_{i_p}$ be a reduced expression for w and define $T_w = T_{i_1} \cdots T_{i_p}$. Then T_w is independent of the choice of the reduced expression of w and*

$$T_{w^{-1}} [\mathcal{O}_{X_1}] = [\mathcal{O}_{X_w}].$$

Proof. The formula in the statement follows from Propositions 3.3 and 3.4. These two Propositions, combined with formula (3.1c) also show that the action of T_w on the elements of the basis $\{[\mathcal{O}_v] \mid v \in W\}$ of $K(G/B)$ is independent of the choice of the reduced word for w . By Proposition 1.5, $K(G/B)$ is a free $R(G)$ -module and thus it follows from (3.1c) that, as an operator on $R(T)$, T_w is independent of the reduced word for w . ■

4. The Pieri-Chevalley formula

In this section we shall inductively apply formula (3.1d) to obtain an expansion of the product $e^\lambda[\mathcal{O}_{X_w}]$ in $K(G/B)$ in terms of the basis $\{[\mathcal{O}_{X_v}] \mid v \in W\}$. We use the path model of P. Littelmann to keep track of the combinatorics involved in iterating formula (3.1d).

The path model

Let $\Lambda = \sum_i \mathbb{Z}\omega_i$ be the weight lattice and let $\mathfrak{h}^* = \sum_i \mathbb{R}\omega_i$. A *path* in \mathfrak{h}^* is a piecewise linear map $\pi: [0, 1] \rightarrow \mathfrak{h}^*$ such that $\pi(0) = 0$. Let π be a path, let α be a simple root and let $h_\alpha: [0, 1] \rightarrow \mathbb{R}$ be the function given by

$$\begin{aligned} h_\alpha: [0, 1] &\longrightarrow \mathbb{R} \\ t &\longmapsto \langle \pi(t), \alpha^\vee \rangle. \end{aligned}$$

At t this function gives the position of $\pi(t)$ in the α -direction. Let m_α be the minimal value of h_α and define functions $l: [0, 1] \rightarrow [0, 1]$ and $r: [0, 1] \rightarrow [0, 1]$ by

$$l(t) = \min\{1, h_\alpha(s) - m_\alpha \mid t \leq s \leq 1\}, \quad r(t) = 1 - \min\{1, h_\alpha(s) - m_\alpha \mid 0 \leq s \leq t\}.$$

The *root operators* (see [L3] Definitions 2.1 and 2.2) are operators on the paths given by

$$e_\alpha \pi = \begin{cases} t \mapsto \pi(t) + r(t)\alpha, & \text{if } r(0) = 0, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and}$$

$$f_\alpha \pi = \begin{cases} t \mapsto \pi(t) - l(t)\alpha, & \text{if } l(1) = 1, \\ 0, & \text{otherwise,} \end{cases}$$

where we use 0 to denote the “null path”.

Fix a dominant weight $\lambda \in \Lambda$. Let π_λ be the path given by $\pi_\lambda(t) = t\lambda$, $0 \leq t \leq 1$, and let

$$\mathcal{T}^\lambda = \{f_{i_1} f_{i_2} \cdots f_{i_\ell} \pi_\lambda\}$$

be the set of all paths obtained by applying sequences of root operators $f_i = f_{\alpha_i}$, $1 \leq i \leq n$ to π_λ . This is the set of Lakshmibai-Seshadri paths of shape λ . P. Littelmann [L1] has shown that this set of paths is finite and can be characterized in terms of an integrality condition. We shall not need this alternative characterization.

Let W_λ be the stabilizer of λ . The cosets in W/W_λ are partially ordered by the Bruhat-Chevalley order. Use a pair of sequences

$$\begin{aligned} \vec{\tau} &= (\tau_1 > \tau_2 > \cdots > \tau_\ell), & \tau_i &\in W/W_\lambda, & \text{and} \\ \vec{a} &= (0 = a_0 < a_1 < a_2 < \cdots < a_\ell = 1), & a_i &\in \mathbb{Q}, \end{aligned}$$

to encode the path $\pi: [0, 1] \rightarrow \mathfrak{h}^*$ given by

$$\pi(t) = (t - a_{j-1})\tau_j\lambda + \sum_{i=1}^{j-1} (a_i - a_{i-1})\tau_i\lambda, \quad \text{for } a_{j-1} \leq t \leq a_j.$$

We shall write $\pi = (\vec{\tau}, \vec{a})$. Every path $\pi \in \mathcal{T}^\lambda$ is of this form. Littelmann introduced this set of paths \mathcal{T}^λ as a model for the Weyl character formula. He proved that

$$\sum_{\eta \in \mathcal{T}^\lambda} e^{\eta(1)} = \frac{\sum_{w \in W} \varepsilon(w) e^{w(\lambda + \rho)}}{\sum_{w \in W} \varepsilon(w) e^{w\rho}},$$

where $\rho = \frac{1}{2} \sum_{\alpha > 0} \alpha$ is the half-sum of the positive roots.

Application of the path model

Fix a dominant weight $\lambda \in \Lambda$ and let $\pi = (\vec{\tau}, \vec{a}) = ((\tau_1 > \cdots > \tau_r), (a_0 < \cdots < a_r)) \in \mathcal{T}^\lambda$. The *initial direction* of π is $\iota(\pi) = \tau_1$. Fix $w \in W$, let $\bar{w} = wW_\lambda \in W/W_\lambda$ and define

$$\mathcal{T}_w^\lambda = \{\pi \in \mathcal{T}^\lambda \mid \iota(\pi) \leq \bar{w}\}.$$

Let $\pi = (\vec{\tau}, \vec{a}) \in \mathcal{T}_w^\lambda$. A *maximal lift of $\vec{\tau}$ with respect to w* is a choice of representatives $t_i \in W$ of the cosets τ_i such that $w \geq t_1 > \cdots > t_r$ and each t_i is maximal in Bruhat order such that $t_{i-1} > t_i$. The *final direction* of π with respect to w is

$$v(\pi, w) = t_r,$$

where $w \geq t_1 > \cdots > t_r$ is a maximal lift of $\tau_1 > \cdots > \tau_r$ with respect to w .

For each $\pi \in \mathcal{T}^\lambda$ such that $e_\alpha(\pi) = 0$ the α -string of π is the set of paths

$$S_\alpha(\pi) = \{\pi, f_\alpha \pi, \dots, f_\alpha^m \pi\},$$

where m is maximal such that $f_\alpha^m \pi \neq 0$. We have:

- (a) If $f_\alpha^j \pi \neq 0$ then $(f_\alpha^j \pi)(1) = \pi(1) - j\alpha$.
- (b) $\iota(f_\alpha^j \pi) = s_\alpha \iota(\pi)$ for all $1 \leq j \leq m$.
- (c) If $S_\alpha(\pi) \subseteq \mathcal{T}_w^\lambda$ then $v(f_\alpha^m \pi, w) = s_\alpha v(\pi, w)$ and $v(f_\alpha^j \pi, w) = v(\pi, w)$ for $1 \leq j < m$.

Statement (a) is [L2] Lemma 2.1a, statement (b) is [L1] Lemma 5.3b, and statement (c) follows from [L1] Lemma 5.3c and [L2] Lemma 2.1e. All of these facts are really coming from the explicit form of the action of the root operators on the Lakshmibai-Seshadri paths which is given in [L1] Proposition 4.2. The consequence of (a) and (c) is that

$$\sum_{\eta \in S_\alpha(\pi)} T_{v(\eta, w)^{-1}} e^{\eta(1)} = T_{v(\pi, w)^{-1}} \left(T_\alpha e^{s_\alpha \pi(1)} + \frac{e^{\pi(1)} - e^{s_\alpha \pi(1)}}{1 - e^{-\alpha}} \right) = T_{v(\pi, w)^{-1}} e^{\pi(1)} T_\alpha.$$

Let $w = s_\alpha w'$ where $\ell(w) = \ell(w') + 1$. Let $\pi \in \mathcal{T}_w^\lambda$ be such that $e_\alpha(\pi) = 0$. It follows from (a) that

$$S_\alpha(\pi) \subseteq \mathcal{T}_w^\lambda, \quad \text{and} \quad \text{either } S_\alpha(\pi) \cap \mathcal{T}_{w'}^\lambda = \{\pi\} \text{ or } S_\alpha(\pi) \subseteq \mathcal{T}_{w'}^\lambda.$$

Suppose that $w \geq t_1 > \cdots > t_r$ and $w' \geq t'_1 > \cdots > t'_r$ are maximal lifts of π with respect to w and w' respectively.

If $m > 0$ then t_1 is not divisible by s_α . It follows that $t'_1 = t_1$ and thus that $t_r = t'_r$.

If $m = 0$ then all the t_i are divisible by s_α and it follows that $t_r = s_\alpha t'_r$.

Thus $v(\pi, w) = v(\pi, w')$ if $m > 0$ and $v(\pi, w) = v(\pi, w') s_\alpha$ if $m = 0$. We conclude that

$$\begin{aligned} T_{v(\pi, w')^{-1}} e^{\pi(1)} T_\alpha &= T_{v(\pi, w')^{-1}} \left(T_\alpha e^{s_\alpha \pi(1)} + \frac{e^{\pi(1)} - e^{s_\alpha \pi(1)}}{1 - e^{-\alpha}} \right) \\ &= \sum_{\eta \in S_\alpha(\pi)} T_{v(\eta, w)^{-1}} e^{\eta(1)}. \end{aligned} \tag{4.1}$$

Theorem 4.2. *Let λ be a dominant integral weight and let $w \in W$. Then*

$$e^\lambda T_{w^{-1}} = \sum_{\eta \in \mathcal{T}_w^\lambda} T_{v(\eta, w)^{-1}} e^{\eta(1)}$$

as operators on $R(T)$.

Proof. The proof is by induction on $\ell(w)$. The base case $\ell(w) = 1$ is formula (3.1d). Let $w = s_\alpha w'$ with $\ell(w) = \ell(w') + 1$. Then

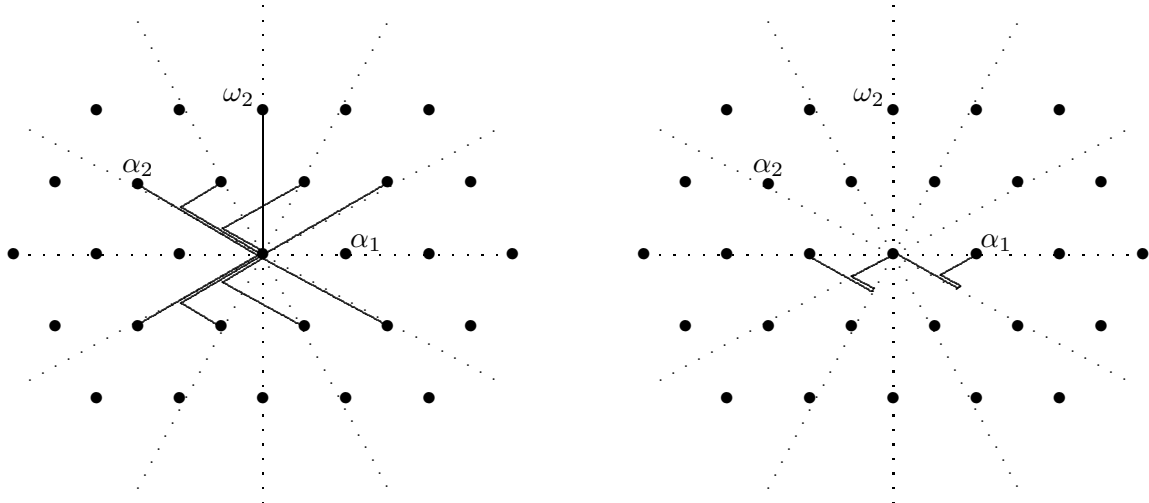
$$\begin{aligned}
e^\lambda T_{w^{-1}} &= e^\lambda T_{(w')^{-1}} T_\alpha \\
&= \left(\sum_{\eta \in \mathcal{T}_{w'}^\lambda} T_{v(\eta, w')^{-1}} e^{\eta(1)} \right) T_\alpha && \text{(by induction)} \\
&= \sum_{\substack{\pi \in \mathcal{T}_w^\lambda \\ e_\alpha(\pi) = 0}} \left(\sum_{S_\alpha(\pi) \subseteq \mathcal{T}_{w'}^\lambda} T_{v(\pi, w')^{-1}} e^{\pi(1)} T_\alpha + \sum_{S_\alpha(\pi) \cap \mathcal{T}_{w'}^\lambda = \{\pi\}} T_{v(\pi, w')^{-1}} e^{\eta(1)} \right) T_\alpha \\
&= \sum_{\substack{\pi \in \mathcal{T}_w^\lambda \\ e_\alpha(\pi) = 0}} \left(\sum_{S_\alpha(\pi) \subseteq \mathcal{T}_{w'}^\lambda} T_{v(\pi, w')^{-1}} e^{\pi(1)} T_\alpha + \sum_{S_\alpha(\pi) \cap \mathcal{T}_{w'}^\lambda = \{\pi\}} T_{v(\pi, w')^{-1}} e^{\eta(1)} T_\alpha \right) \\
&= \sum_{\eta \in \mathcal{T}_w^\lambda} T_{v(\eta, w)^{-1}} e^{\eta(1)} && \text{(by (4.1)).} \quad \blacksquare
\end{aligned}$$

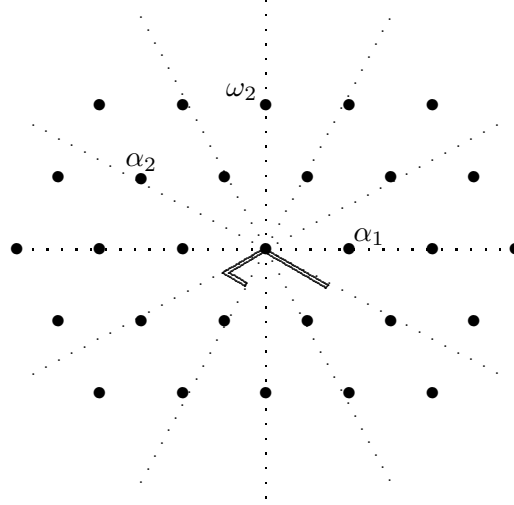
Theorem 4.3. *Let λ be a dominant integral weight. In $K(G/B)$*

$$e^\lambda [\mathcal{O}_{X_w}] = \sum_{\eta \in \mathcal{T}_w^\lambda} [\mathcal{O}_{X_{v(\eta, w)}}]$$

Proof. Since the sheaf \mathcal{O}_{X_1} is supported on the single point $X_1 \in G/B$, any product $[\mathcal{F}][\mathcal{O}_{X_1}]$, where \mathcal{F} is a vector bundle on G/B is the class of a bundle supported on the single point X_1 . More precisely, $[\mathcal{F}][\mathcal{O}_{X_1}] = \text{rk}(\mathcal{F})[\mathcal{O}_{X_1}]$. Thus, since e^λ is the class of a line bundle we have $e^\lambda[\mathcal{O}_{X_1}] = [\mathcal{O}_{X_1}]$. By Corollary 3.6, $[\mathcal{O}_{X_w}] = T_{w^{-1}}[\mathcal{O}_{X_1}]$ and so the result follows from Theorem 4.2. \blacksquare

The following example illustrates how one computes the product $e^{\omega_2}[\mathcal{O}_{X_{s_1 s_2 s_1 s_2}}]$ in $K(G/B)$ for the group G of type G_2 . In this case $\lambda = \omega_2$, $w^{-1} = s_2 s_1 s_2 s_1$ and the starting path π_λ is the straight line path from the origin to the point ω_2 . The paths in the set $\mathcal{T}_{s_2 s_1 s_2 s_1}^{\omega_2}$ are the paths in the following diagrams.





These paths yield the following data:

endpoint	maximal lift \vec{t}	$\iota(\eta) = \tau_1$	$v(w, \eta)^{-1}$
ω_2	(s_1)	$\bar{1}$	s_1
$s_2\omega_2$	(s_2s_1)	$\bar{s_2}$	s_1s_2
$s_2\omega_2 - \alpha_1$	$(s_1s_2s_1 > s_2s_1)$	$\overline{s_1s_2}$	s_1s_2
$s_2\omega_2 - 2\alpha_1$	$(s_1s_2s_1 > s_2s_1)$	$\overline{s_1s_2}$	s_1s_2
$s_1s_2\omega_2$	$(s_1s_2s_1)$	$\overline{s_1s_2}$	$s_1s_2s_1$
$s_2s_1s_2\omega_2$	$(s_2s_1s_2)$	$\overline{s_2s_1s_2}$	$s_2s_1s_2$
$s_2s_1s_2\omega_2 - \alpha_1$	$(s_1s_2s_1s_2 > s_2s_1s_2)$	$\overline{s_1s_2s_1s_2}$	$s_2s_1s_2$
$s_2s_1s_2\omega_2 - 2\alpha_1$	$(s_1s_2s_1s_2 > s_2s_1s_2)$	$\overline{s_1s_2s_1s_2}$	$s_2s_1s_2$
$s_1s_2s_1s_2\omega_2$	$(s_1s_2s_1s_2)$	$\overline{s_1s_2s_1s_2}$	$s_2s_1s_2s_1$
α_1	$(s_2s_1s_2 > s_1s_2 > s_2)$	$\overline{s_2s_1s_2}$	s_2
$-\alpha_1$	$(s_1s_2s_1s_2 > s_2s_1s_2 > s_1s_2)$	$\overline{s_1s_2s_1s_2}$	s_2s_1
0	$(s_2s_1s_2 > s_1s_2)$	$\overline{s_2s_1s_2}$	s_2s_1
0	$(s_1s_2s_1s_2 > s_2s_1s_2 > s_1s_2 > s_2)$	$\overline{s_1s_2s_1s_2}$	s_2

and thus we get

$$\begin{aligned}
e^{\omega_2} T_{s_2s_1s_2s_1} &= T_{s_1} e^{\omega_2} + T_{s_1s_2} e^{s_2\omega_2} + T_{s_1s_2} e^{s_2\omega_2 - \alpha_1} + T_{s_1s_2} e^{s_2\omega_2 - 2\alpha_1} + T_{s_1s_2s_1} e^{s_1s_2\omega_2} \\
&\quad + T_{s_2s_1s_2} e^{s_1s_2s_1\omega_2} + T_{s_2s_1s_2} e^{s_2s_1s_2\omega_2 - \alpha_1} + T_{s_2s_1s_2} e^{s_2s_1s_2\omega_2 - 2\alpha_1} \\
&\quad + T_{s_2s_1s_2s_1} e^{s_1s_2s_1s_2\omega_2} + T_{s_2} e^{\alpha_1} + T_{s_2s_1} e^{-\alpha_1} + T_{s_2s_1} e^0 + T_{s_2} e^0
\end{aligned}$$

and

$$e^{\omega_2} [\mathcal{O}_{X_{s_1s_2s_1s_2}}] = [\mathcal{O}_{X_{s_1s_2s_1s_2}}] + [\mathcal{O}_{X_{s_1s_2s_1}}] + 3[\mathcal{O}_{X_{s_2s_1s_2}}] + 3[\mathcal{O}_{X_{s_2s_1}}] + 2[\mathcal{O}_{X_{s_1s_2}}] + 2[\mathcal{O}_{X_{s_2}}] + [\mathcal{O}_{X_{s_1}}].$$

5. Passage to $H^*(G/B)$

Let us explain how our results in $K(G/B)$ are related to the cohomology $H^*(G/B)$ and Schubert polynomials. The transfer is by way of the Chern character ch .

If X is a finite CW complex then the *Chern character* (see [Mac, Ch. 10], [Hi, §23-24], [Ha, App. A])

$$\text{ch}: K_{vb}(X) \otimes \mathbb{Q} \longrightarrow H^*(X; \mathbb{Q})$$

is a natural ring isomorphism, i.e. if $f: X \rightarrow Y$ is continuous then $\text{ch}(f^!(x)) = f^*(\text{ch}(x))$. If $f: X \rightarrow Y$ is a morphism of nonsingular projective varieties then the Grothendieck-Riemann-Roch theorem [Ha, App. A Theorem 5.3], henceforth G-R-R, says

$$\text{ch}(f_!(x)) = f_*(\text{ch}(x)\text{td}(\mathcal{T}_f)),$$

where $\text{td}(\mathcal{T}_f)$ is Todd class of the relative tangent sheaf of f . If \mathcal{L} is a line bundle on X with first Chern class $\lambda \in H^2(X; \mathbb{Q})$ then the Chern character and the Todd class of \mathcal{L} are the elements of $H^*(X)$ given by

$$\text{ch}(\mathcal{L}) = e^\lambda = \sum_{k \geq 0} \frac{\lambda^k}{k!} \quad \text{and} \quad \text{td}(\mathcal{L}) = \frac{\lambda}{1 - e^{-\lambda}}, \quad \text{respectively.}$$

The expression e^λ is a finite sum since $\lambda^k = 0$ in $H^*(X)$ whenever $k > \dim(X)$.

$H^(G/B)$ as the quotient of a polynomial ring*

Let $X = G/B$. Let \mathfrak{h} be the Lie algebra of T and let $S(\mathfrak{h}^*)$ be the ring of polynomials on \mathfrak{h}^* (over \mathbb{Q}). This is a polynomial ring in the n variables $\alpha_1, \dots, \alpha_n$ (the simple roots). Let $\hat{\varepsilon}: S(\mathfrak{h}^*) \rightarrow \mathbb{Q}$ be the homomorphism given by $\hat{\varepsilon}(\lambda) = 0$ for all $\lambda \in \mathfrak{h}^*$. If $f \in S(\mathfrak{h}^*)$ then $\hat{\varepsilon}(f)$ is the constant term of f . It is a classical theorem of Borel (see [BG, Prop. 1.3]) that

$$H^*(G/B; \mathbb{Q}) \cong \frac{S(\mathfrak{h}^*)}{\hat{\mathcal{I}}}, \quad (5.1)$$

where $\hat{\mathcal{I}}$ is the ideal of $S(\mathfrak{h}^*)$ generated by $\{f \in S(\mathfrak{h}^*)^W \mid f - \hat{\varepsilon}(f) = 0\}$. Proposition 1.5 is the K-theory analogue of Borel's theorem.

Let $-\lambda \in \Lambda$. The element $-\lambda$ determines a character of T , denoted by $e^\lambda \in R(T)$ (see section 1). Let $c_1(\mathcal{L}_{-\lambda}) \in H^2(G/B)$ be the first Chern class of the line bundle $\mathcal{L}_{-\lambda} = \phi(e^\lambda)$ where $\phi: R(T) \rightarrow K(G/B)$ is the map in (1.2). Because of the isomorphism in (5.1) we often abuse notation and write $\lambda = c_1(\mathcal{L}_{-\lambda}) \in H^*(G/B)$. All of the maps in the following commutative diagram are isomorphisms. (Recall that we can identify $K(G/B)$ and $K_{vb}(G/B)$.)

$$\begin{array}{ccc} R(T)/\mathcal{I} \otimes \mathbb{Q} & \xrightarrow{\phi} & K_{vb}(G/B) \otimes \mathbb{Q} \\ \downarrow \text{ch} & & \downarrow \text{ch} \\ S(\mathfrak{h}^*)/\hat{\mathcal{I}} & \xrightarrow{\hat{\phi}} & H^*(G/B; \mathbb{Q}) \end{array} \quad \text{where} \quad \begin{array}{ccc} e^\lambda & \xrightarrow{\phi} & [\mathcal{L}_{-\lambda}] \\ \downarrow \text{ch} & & \downarrow \text{ch} \\ e^\lambda & \xrightarrow{\hat{\phi}} & e^{c_1(\mathcal{L}_{-\lambda})} \end{array} \quad (5.2)$$

The left hand ch map is obtained by viewing $R(T)$ and $S(\mathfrak{h}^*)$ as subsets of $K(\mathcal{B}_T)$ and $\widehat{H}^*(\mathcal{B}_T)$, respectively, where \mathcal{B}_T is the classifying space of T .

The relation between T_α and ∂_α

Let α be a simple root and let P_α be the parabolic subgroup with Lie algebra \mathfrak{p}_α spanned by \mathfrak{b} and the root space $\mathfrak{g}_{-\alpha}$. Let $f_\alpha: G/B \rightarrow G/P_\alpha$ corresponding \mathbb{P}_1 -bundle.

A proof of Proposition 3.3: Since $K(G/B)$ is torsion free, we can check the identity $f_\alpha^!(f_\alpha)_!(x) = T_\alpha(x)$ by applying ch to both sides and checking the result in cohomology.

The G-R-R for the map f_α says

$$\text{ch}((f_\alpha)_!(x)) = (f_\alpha)_*(\text{ch}(x)\text{td}(\mathcal{T}_{f_\alpha})), \quad (5.3)$$

where \mathcal{T}_{f_α} is the bundle of tangents along the fibres, which is the line bundle associated to the $\text{ad}(B)$ -module $\mathfrak{p}_\alpha/\mathfrak{b}$. Since this module has weight $-\alpha$, $c_1(\mathcal{T}_{f_\alpha}) = \alpha \in H^2(G/B; \mathbb{Z})$ and so (5.3) becomes

$$\text{ch}((f_\alpha)!(x)) = (f_\alpha)_* \left(\text{ch}(x) \frac{\alpha}{1 - e^{-\alpha}} \right). \quad (5.4)$$

The BGG operator $\partial_\alpha = (f_\alpha)^*(f_\alpha)_*$ is explicitly given by the formula

$$\partial_\alpha(z) = \frac{z - s_\alpha(z)}{\alpha}, \quad (5.5)$$

see [D]. Thus, by applying $(f_\alpha)^*$ to both sides of (5.4) we obtain

$$\text{ch}((f_\alpha)!(f_\alpha)!(x)) = (f_\alpha)^*(f_\alpha)_* \left(\text{ch}(x) \frac{\alpha}{1 - e^{-\alpha}} \right) = \partial_\alpha \left(\text{ch}(x) \frac{\alpha}{1 - e^{-\alpha}} \right). \quad (5.6)$$

The strategy now is to manipulate the right hand side of (5.6) using the “skew-Leibniz” rule satisfied by ∂_α to obtain $\text{ch}(T_\alpha(x))$. For convenience, let $y = \text{ch}(x)$. Then the right side of (5.6) is

$$\partial_\alpha \left(y \frac{\alpha}{1 - e^{-\alpha}} \right) = y \partial_\alpha \left(\frac{\alpha}{1 - e^{-\alpha}} \right) + s_\alpha \left(\frac{\alpha}{1 - e^{-\alpha}} \right) \partial_\alpha(y)$$

and we claim

$$(a) \quad s_\alpha \left(\frac{\alpha}{1 - e^{-\alpha}} \right) = \frac{\alpha}{e^\alpha - 1}, \quad (b) \quad \partial_\alpha \left(\frac{\alpha}{1 - e^{-\alpha}} \right) = 1.$$

The first equality is trivial and the second can be proved by formal computation (carefully done!) or by applying the G-R-R again. Using (a) and (b) we obtain

$$\partial_\alpha \left(y \frac{\alpha}{1 - e^{-\alpha}} \right) = y + \frac{\alpha}{e^\alpha - 1} \left(\frac{y - s_\alpha(y)}{\alpha} \right).$$

Now cancelling the α 's in the second term on the right and recalling that $y = \text{ch}(x)$ we find

$$\text{ch}((f_\alpha)!(f_\alpha)!(x)) = \text{ch}(T_\alpha(x)). \quad \blacksquare$$

We see that

$$\text{ch}(T_\alpha(x)) = \partial_\alpha \left(\text{ch}(x) \frac{\alpha}{1 - e^{-\alpha}} \right) \quad (5.7)$$

which relates T_α to ∂_α in an explicit way. In fact, if one wished one could reverse the argument and derive the formula (5.5) for ∂_α from the formula for T_α .

A proof of Proposition 1.6

Proposition 1.6. *If $f: G/B \rightarrow G/P$ is the natural projection then the induced map $f^!: K(G/P) \rightarrow K(G/B)$ is an injection.*

Proof. Since $K(G/P)$ is torsion free there is a natural injection $K(G/P) \hookrightarrow K(G/P) \otimes \mathbb{Q} \rightarrow H^*(G/P; \mathbb{Q})$ and so it suffices to check that the pull-back f^* in rational cohomology is injective. Since the odd cohomology groups of the base and fiber are zero the Serre spectral sequence of the bundle $P/B \rightarrow G/B \rightarrow G/P$ shows that f^* is injective even for $H^*(G/P; \mathbb{Z})$. \blacksquare

Dictionary between $K(G/B)$ and $H^(G/B)$*

In summary, the Chern character gives an isomorphism

$$\begin{array}{ccccc} R(T)/\mathcal{I} & \cong & K(G/B) & \xrightarrow{\text{ch}} & H^*(G/B) & \cong & S(\mathfrak{h}^*)/\hat{\mathcal{I}} \\ & & e^\lambda & \longmapsto & e^\lambda, & & \end{array}$$

where

$$\mathcal{I} = \text{ideal generated by } \{f \in R(T)^W \mid f - \varepsilon(f) = 0\}, \quad \varepsilon: \begin{array}{ccc} R(T) & \longrightarrow & \mathbb{Z} \\ e^\lambda & \longmapsto & 1, \end{array}$$

$$\hat{\mathcal{I}} = \text{ideal generated by } \{f \in S(\mathfrak{h}^*)^W \mid f - \hat{\varepsilon}(f) = 0\}, \quad \hat{\varepsilon}: \begin{array}{ccc} S(\mathfrak{h}^*) & \longrightarrow & \mathbb{Z} \\ \lambda & \longmapsto & 0. \end{array}$$

Let $[X_w] \in H^*(G/B)$ be the element which is Poincaré dual to the fundamental cycle of X_w in $H_*(G/B)$. This element is called a *Schubert polynomial*. Then

$$K(G/B) \text{ has basis } \{[\mathcal{O}_{X_w}] \mid w \in W\} \quad \text{and} \quad H^*(G/B) \text{ has basis } \{[X_w] \mid w \in W\}.$$

From a general fact (see [Fu, Ex. 15.2.16] or [CG, 5.8.13(i) and p. 289])

$$\text{ch}([\mathcal{O}_{X_w}]) = [X_w] + \text{higher degree terms.}$$

where $\deg([X_w]) = \dim(G/B) - \dim(X_w) = N - \ell(w)$, where N is the number of positive roots for \mathfrak{g} .

If α is a simple root and $f_\alpha: G/B \rightarrow G/P_\alpha$ is the corresponding \mathbb{P}^1 -bundle, then

$$T_\alpha(x) = (f_\alpha)^!(f_\alpha)_!(x) = \frac{e^\alpha x - s_\alpha(x)}{e^\alpha - 1} \quad \text{and} \quad \partial_\alpha(x) = (f_\alpha)^*(f_\alpha)_*(x) = \frac{x - s_\alpha(x)}{\alpha}$$

in $K(G/B)$ and $H^*(G/B)$, respectively. As illustrated in (5.7) each of these two formulas can be derived from the other via the use of the Grothendieck-Riemann-Roch Theorem. This means that the following formulas (Proposition 3.4 and [BGG, Th. 3.14])

$$T_\alpha([\mathcal{O}_{X_w}]) = \begin{cases} [\mathcal{O}_{X_{ws_\alpha}}], & \text{if } ws_\alpha > w, \\ [\mathcal{O}_{X_w}], & \text{if } ws_\alpha < w, \end{cases} \quad \text{and} \quad \partial_\alpha([X_w]) = \begin{cases} [X_{ws_\alpha}], & \text{if } ws_\alpha > w, \\ [X_w], & \text{if } ws_\alpha < w, \end{cases}$$

are equivalent.

Our new Pieri-Chevalley formula in $K(G/B)$, Theorem 4.3, and Chevalley's classical Pieri formula in $H^*(G/B)$, [Ch, Prop. 10], are

$$e^\lambda[\mathcal{O}_{X_w}] = \sum_{\eta \in \mathcal{T}_w^\lambda} [\mathcal{O}_{X_{v(\eta, w)}}] \quad \text{and} \quad \lambda \cdot [X_w] = \sum_{v \xrightarrow{\alpha} w} \langle \lambda, \alpha^\vee \rangle [X_v],$$

where the sum is over all $v \in W$ such that $\ell(v) = \ell(w) - 1$ and there is a root α such that $v = s_\alpha w$. Chevalley's formula can be obtained from ours formula by subtracting $[\mathcal{O}_{X_w}]$ from each side, applying the Chern character ch , and comparing the lowest degree terms on each side.

REFERENCES

- [BGG] I.N. BERNSTEIN, I.M. GEL'FAND AND S.I. GEL'FAND, *Schubert cell and cohomology of the spaces G/P* , Russ. Math. Surv. **28** (3) (1973), 1–26.
- [BS] A. BOREL AND J.-P. SERRE, *Le Théorème de Riemann-Roch (d'après Grothendieck)*, Bull. Soc. Math. France **68** (1958), 97–136.
- [C] C. CHEVALLEY, *Le classes d'équivalence rationnelle*, I, II, Seminaire C. Chevalley, Anneaux de Chow et applications (mimeographed notes), Paris, 1958.
- [Ch] C. CHEVALLEY, *Sur les decompositions cellulaires des espaces G/B* , in *Algebraic Groups and their Generalizations: Classical Methods*, W. Haboush and B. Parshall eds., Proc. Symp. Pure Math., Vol. **56** Pt. 1, Amer. Math. Soc. (1994), 1–23.
- [CG] N. CHRISS AND V. GINZBURG, *Representation theory and complex geometry*, Birkhäuser, Boston, 1997.
- [D] M. DEMAZURE, *Désingularisation des variétés de Schubert généralisées*, Ann. Sci. École Norm. Sup. **7** (1974), 53–88.
- [FL] W. FULTON AND A. LASCOUX, *A Pieri formula in the Grothendieck ring of a flag bundle*, Duke Math. J. **76** (1994), 711–729.
- [Fu] W. FULTON, *Intersection Theory*, Ergebnisse der Mathematik (3) **2**, Springer-Verlag, Berlin-New York, 1984.
- [Ha] R. HARTSHORNE, *Algebraic geometry*, Graduate Texts in Mathematics **52**, Springer-Verlag, New York-Heidelberg, 1977.
- [H] F. HIRZEBRUCH, *Topological methods in algebraic geometry*, Classics in Mathematics, Springer-Verlag, Berlin, 1995.
- [Ke] G. KEMPF, *Linear systems on homogeneous spaces*, Ann. Math. **103** (1976), 557–591.
- [KK] B. KOSTANT AND S. KUMAR, *T-equivariant K-theory of generalized flag varieties*, J. Differential Geom. **32** (1990), 549–603.
- [L1] P. LITTELMANN, *A Littlewood-Richardson rule for symmetrizable Kac-Moody algebras*, Invent. Math. **116** (1994), 329–346.
- [L2] P. LITTELMANN, *Paths and root operators in representation theory*, Ann. Math. **142** (1995), 499–525.
- [L3] P. LITTELMANN, *Characters of representations and paths in $\mathfrak{S}_{\mathbb{R}}^*$* , Proc. Symp. Pure Math. **61** (1997), 29–49.
- [Mac] I.G. MACDONALD, *Algebraic geometry: Introduction to schemes*, W. A. Benjamin, New York-Amsterdam, 1968.
- [P] H. PITTIE, *Homogeneous vector bundles over homogeneous spaces*, Topology **11** (1972), 199–203.
- [Ra] A. RAMANATHAN, *Frobenius splitting and Schubert varieties*, in *Proceedings of the Hyderabad Conference on Algebraic Groups* (Hyderabad, 1989), Manoj Prakashan, Madras, 1991, p. 497–508.
- [W] A. WEIL, *Démonstration topologique d'un théorème fondamental de Cartan*, C.R. Acad. Sci. **200** (1935), 518–520.